Extension of the Buchalla–Safir bound

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Abstract

I provide a simple derivation of the Buchalla–Safir bound on γ. I generalize it to the case where an upper bound on the phase of the penguin pollution is assumed. I apply the Buchalla–Safir bound, and its generalization, to the recent Belle data on CP violation in $B \to \pi^+\pi^-$.

1 Introduction

CP violation in B_d^0–$\bar{B}_d^0$ mixing and in the decays of those mesons to $\pi^+\pi^-$ is parametrized by

$$\lambda = \frac{q}{p} \frac{\bar{A}}{A}, \tag{1}$$

where q/p relates to B_d^0–$\bar{B}_d^0$ mixing, A is the amplitude for $B_d^0 \to \pi^+\pi^-$, and $\bar{A}$ is the amplitude for $\bar{B}_d^0 \to \pi^+\pi^-$ [1]. Two CP-violating quantities can be measured:

$$S = \frac{2 \mathrm{Im}\, \lambda}{1 + |\lambda|^2}, \tag{2}$$

$$C = \frac{1 - |\lambda|^2}{1 + |\lambda|^2}. \tag{3}$$

Let

$$\frac{q}{p} = \exp\left(-2i\tilde{\beta}\right). \tag{4}$$

In the Standard Model (SM), $\tilde{\beta} = \beta$ and the sine of 2β is measured [2] through CP violation in $B_d^0/\bar{B}_d^0 \to \psi K_S$:

$$\sin 2\beta = 0.736 \pm 0.049. \tag{5}$$

In the SM β must be smaller than $\pi/4$, hence $\cos 2\beta$ is assumed positive.

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Together with eq. (4), I shall assume that, as in the SM,

$$\frac{\bar{A}}{A} = \frac{e^{-i\gamma} + z}{e^{i\gamma} + z}, \tag{6}$$

where γ is another CP-violating phase, which we would like to measure too. In the SM, $0 \leq \gamma \leq \pi - \beta$. The parameter z represents the 'penguin pollution', an annoying contribution from penguin diagrams which we must somehow circumvent if we want to get at γ.

Buchalla and Safir (BS) [3] have found a solution to the following problem. Suppose that

- one has measured $\sin 2\tilde{\beta}$ and S,
- one has found that $S > -\sin 2\tilde{\beta}$,
- one assumes the validity of the SM, and
- one assumes that $\mathrm{Re}\, z > 0$.

Is it then possible to find a lower bound on γ stronger than $\gamma \geq 0$? The solution to this problem, as given by BS, is

$$\gamma > \frac{\pi}{2} - \arctan \frac{S - \tau + \tau\sqrt{1 - S^2}}{\tau S + 1 - \sqrt{1 - S^2}}, \tag{7}$$

where

$$\tau \equiv \frac{\sin 2\tilde{\beta}}{1 - \sqrt{1 - \sin^2 2\tilde{\beta}}} \tag{8}$$

and the square roots in eqs. (7) and (8) are, by definition, positive.

In this Letter I provide a simple derivation of the BS bound, which does not rely on any assumptions about the quark mixing matrix. I also consider the realistic situation where both S and C have been measured; this allows one to put a stronger bound on γ than when one knows only S. Inspired by the result, quoted by BS, of a computation of z yielding

$$\arg z = 0.15 \pm 0.25, \tag{9}$$

I furthermore consider the situation where one assumes an upper bound on $|\arg z|$. Finally, I apply the BS bound, and its extensions, to the most recent measurements of S and C made public by the experimental collaboration Belle [4].

2 The Buchalla–Safir bound

I define

$$\begin{aligned} x &\equiv \lambda \exp\left(2i\tilde{\beta}\right) \\ &= \frac{e^{-i\gamma} + z}{e^{i\gamma} + z}. \end{aligned} \tag{10}$$

Then,

$$C = \frac{1 - |x|^2}{1 + |x|^2}, \tag{11}$$

and I furthermore define

$$\begin{aligned} I &\equiv \frac{2\mathrm{Im}\, x}{1 + |x|^2}, &(12)\\ F &\equiv \frac{|1 - x|^2}{1 + |x|^2} \\ &= 1 - \frac{2\mathrm{Re}\, x}{1 + |x|^2}. &(13) \end{aligned}$$

Clearly,

$$0 \le F \le 2 \tag{14}$$

and

$$C^2 + I^2 + F^2 = 2F. \tag{15}$$

Solving eq. (10) for z, one finds

$$z = -\cos\gamma + \frac{-I + iC}{F} \sin\gamma. \tag{16}$$

Equation (16) has an indeterminacy at the singular point $C = I = F = 0 \Leftrightarrow x = 1$, *i.e.* when $\sin\gamma = 0$, for arbitrary z.

From eq. (16) it follows in particular that

$$F\left(\cos\gamma + \mathrm{Re}\, z\right) + I \sin\gamma = 0. \tag{17}$$

Equation (17) has been first written down by Botella and Silva [5]. It leads to the bound

$$|\mathrm{Re}\, z| \le \frac{\sqrt{F^2 + I^2}}{F}, \tag{18}$$

where $\sqrt{F^2 + I^2}$ *is positive by definition.* The solution to eq. (17) may be written in the form

$$\gamma = \xi + \chi, \tag{19}$$

where (by definition)

- ξ is independent of $\mathrm{Re}\, z$, and
- $\chi = 0$ or $\chi = \pi$ when $\mathrm{Re}\, z = 0$.

One finds

$$\begin{aligned} \cos\xi &= \frac{-I}{\sqrt{F^2 + I^2}}, &(20)\\ \sin\xi &= \frac{F}{\sqrt{F^2 + I^2}}, &(21) \end{aligned}$$

and

$$\sin\chi = \frac{F \mathrm{Re}\, z}{\sqrt{F^2 + I^2}}. \tag{22}$$

While ξ is perfectly defined by eqs. (20) and (21), χ as given by eq. (22) suffers from the twofold ambiguity

$$\chi \rightarrow \pi - \chi. \tag{23}$$

Assuming, as Buchalla and Safir have done, that $\mathrm{Re}\, z > 0$, we see from eqs. (21) and (22) that both ξ and χ are angles either of the first or of the second quadrant. The Buchalla–Safir condition $\mathrm{Re}\, z > 0$ implies the lower bound on γ

$$\begin{aligned} \gamma &> \xi \\ &= \arccos \frac{-I}{\sqrt{F^2 + I^2}}, \end{aligned} \tag{24}$$

together with $\gamma < \xi + \pi$ too. Notice that

$$d\xi = \frac{F dI - I dF}{F^2 + I^2}. \tag{25}$$

Equation (24) does provide a lower bound on γ but, unfortunately, one has to deal with discrete ambiguities. These occur because we are able to measure C but unable to measure I and F; rather, we only know $\sin 2\tilde{\beta}$ and S. Now,

$$I = \frac{2\mathrm{Re}\, \lambda}{1 + |\lambda|^2} \sin 2\tilde{\beta} + S \cos 2\tilde{\beta}, \tag{26}$$

$$F = 1 - \frac{2\mathrm{Re}\, \lambda}{1 + |\lambda|^2} \cos 2\tilde{\beta} + S \sin 2\tilde{\beta}. \tag{27}$$

Assuming that $\sin 2\tilde{\beta}$, S, and C are known, there is a fourfold ambiguity in I and F, since the signs of

$$\frac{2\mathrm{Re}\, \lambda}{1 + |\lambda|^2} = \sqrt{1 - C^2 - S^2}, \tag{28}$$

$$\cos 2\tilde{\beta} = \sqrt{1 - \sin^2 2\tilde{\beta}} \tag{29}$$

remain unknown. Using eqs. (25)–(29),

$$\frac{d\xi}{dC^2} = \frac{-S - \sin 2\tilde{\beta}}{2\left(F^2 + I^2\right)\sqrt{1 - C^2 - S^2}}. \tag{30}$$

(Remember that the sign of $\sqrt{1 - C^2 - S^2}$ is, for the moment, arbitrary.)

Thus, given C, S, and $\sin 2\tilde{\beta}$, there are in reality four different angles ξ:

- ξ_1, in which both $\sqrt{1 - C^2 - S^2}$ and $\cos 2\tilde{\beta}$ are positive,
- ξ_2, in which $\cos 2\tilde{\beta}$ is positive but $\sqrt{1 - C^2 - S^2}$ is negative,

- ξ_3, in which both $\sqrt{1-C^2-S^2}$ and $\cos 2\tilde{\beta}$ are negative, and
- ξ_4, in which $\sqrt{1-C^2-S^2}$ is positive but $\cos 2\tilde{\beta}$ is negative.

Since F remains invariant, and I changes sign, when $\sqrt{1-C^2-S^2}$ and $\cos 2\tilde{\beta}$ change sign simultaneously, we find that $\xi_3 = \pi - \xi_1$ and $\xi_4 = \pi - \xi_2$. From the assumption that $\mathrm{Re}\, z > 0$, and taking into account the indeterminacy in the signs of $\sqrt{1-C^2-S^2}$ and $\cos 2\tilde{\beta}$, one can only deduce that γ must lie in between ξ_k and $\xi_k + \pi$ for all $k = 1, 2, 3$, and 4.

Let us now assume, with BS, the validity of the SM. Then $\cos 2\tilde{\beta}$ is positive and only the values ξ_1 and ξ_2 are allowed for ξ. This produces the lower bound

$$\gamma > \min\left(\xi_1, \xi_2\right). \tag{31}$$

This lower bound is valid in the SM when C, S, and $\sin 2\tilde{\beta}$ are known. It still depends on C^2, since ξ_1 and ξ_2 contain $\sqrt{1-C^2-S^2}$. Consideration of eq. (30), however, shows that, when $S > -\sin 2\tilde{\beta}$, ξ_1 decreases and ξ_2 increases with increasing C^2. Moreover, at the maximum allowed value of C^2, *i.e.* when $C^2 = 1 - S^2$, one has $\xi_1 = \xi_2$, since in general ξ_1 and ξ_2 only differ through the sign of $\sqrt{1-C^2-S^2}$, and that square root becomes zero when $C^2 = 1 - S^2$. This immeadiately leads to the BS bound: if $S > -\sin 2\tilde{\beta}$, then $\gamma > \xi_2\left(C^2 = 0\right)$. It can be shown [5] that, though different in appearance, this bound coincides with the one in eq. (7).

One thus concludes that, if one assumes that $\cos 2\tilde{\beta} > 0$, then

$$\left\{ \begin{array}{lcl} \gamma > \xi_2\left(C^2 = 0\right) & \Leftarrow & S > -\sin 2\tilde{\beta}, \\ \gamma > \xi_1\left(C^2 = 0\right) & \Leftarrow & S < -\sin 2\tilde{\beta}. \end{array} \right. \tag{32}$$

This may be put in a more transparent way if one defines

$$\varphi \equiv \frac{1}{2} \arcsin S, \tag{33}$$

$$\alpha \equiv \pi - \tilde{\beta} - \gamma. \tag{34}$$

The lower bound on γ may then be rewritten as an upper bound on α:

$$\left\{ \begin{array}{lcl} \alpha < \dfrac{\pi}{2} - \varphi & \Leftarrow & \varphi > -\tilde{\beta}, \\ \alpha < \pi + \varphi & \Leftarrow & \varphi < -\tilde{\beta}. \end{array} \right. \tag{35}$$

The discontinuity of the bound at $\varphi = -\tilde{\beta}$ should not come as a surprise. The point $C = 0$, $S = -\sin 2\tilde{\beta}$ allows the singularity $C = I = F = 0$ referred to earlier. When $C = I = F = 0$, γ may be either 0 or π, independently of any assumption on z. Therefore no lower bound on γ may be derived if the experimentally allowed region for C and S includes that point.

It should be stressed that this derivation of the Buchalla–Safir bound on γ, or on α, contains basically no physical assumptions. Only eqs. (1)–(4) and (6), together with $\cos 2\tilde{\beta} > 0$ and $\mathrm{Re}\, z > 0$, are assumed. No assumptions are needed about the physics contained in the decay amplitudes, about the quark mixing matrix, or, indeed, about

anything else; the sole crucial assumption is $\mathrm{Re}\, z > 0$. The Buchalla–Safir bound is purely algebraic.

I now return to the general case where one does not assume the SM. Then, γ may be either positive or negative and, from the assumption that $\mathrm{Re}\, z > 0$, it is only possible to produce a lower bound on $|\gamma|$, never on γ itself. Indeed, given the fourfold ambiguity in the determination of F and I, and the twofold ambiguity in the determination of χ—see eq. (23)—there are eight solutions to eq. (17) for γ. Since, when $\sqrt{1-C^2-S^2}$ and $\cos 2\tilde{\beta}$ change sign simultaneously, I changes sign while F does not change, it is obvious from eq. (17) that those eight solutions pair in four sets through the transformation $\gamma \to -\gamma$. Therefore, only a bound on $|\gamma|$ is possible. Now, computing

$$\tan^2 \xi_1 \left(C^2 = 0 \right) - \tan^2 \xi_2 \left(C^2 = 0 \right) = \frac{-4\sqrt{1-S^2}\sqrt{1-\sin^2 2\tilde{\beta}}}{\left(\sin 2\tilde{\beta} - S \right)^2}, \tag{36}$$

where the square roots in the right-hand side are positive by definition, one finds that $|\tan \xi_1 \left(C^2 = 0 \right)|$ is always smaller than $|\tan \xi_2 \left(C^2 = 0 \right)|$. Hence,

$$|\gamma| > \arctan \left| \tan \xi_1 \left(C^2 = 0 \right) \right|. \tag{37}$$

Using again φ as defined in eq. (33), one concludes that

$$|\gamma| > \left| \tilde{\beta} + \varphi \right|, \tag{38}$$

which is valid in any model provided $\mathrm{Re}\, z > 0$—and provided the basic equations (1)–(4) and (6) hold, of course.

3 Assuming an upper limit on $|\arg z|$

In their work [3], Buchalla and Safir have quoted the result of a computation (in the context of the Standard Model) of z as yielding the result in eq. (9). They have thereby justified their assumption $\mathrm{Re}\, z > 0$. In this section I shall consider a different assumption,

$$|\cot \arg z| > L, \tag{39}$$

where L is some positive number. Clearly, this assumption is complementary to $\mathrm{Re}\, z > 0$; while $\mathrm{Re}\, z > 0$, by itself alone, leaves $\cot \arg z$ completely arbitrary, eq. (39), by itself alone, does not provide any information on whether $\mathrm{Re}\, z$ is positive or negative. If L is, for instance, taken equal to 1, then eq. (39) is well justified by eq. (9).

In order to find the consequences of the assumption in eq. (39), I return to eq. (16) and therefrom derive that

$$C \cot \arg z + F \cot \gamma + I = 0. \tag{40}$$

Hence,

$$|\cot \arg z| > L \quad \Leftrightarrow \quad \cot \gamma < \frac{-I - L\,|C|}{F} \quad \text{or} \quad \cot \gamma > \frac{-I + L\,|C|}{F}. \tag{41}$$

Clearly, this condition makes smaller the range for γ allowed by $\mathrm{Re}\, z > 0$ alone; that range, remember, is given by $\xi < \gamma < \xi + \pi$, where ξ belongs either to the first or to the second quadrant and $\cot \xi = -I/F$.

Let us now assume the validity of the SM. Then $\gamma \leq \pi - \beta$ and the relevant bound on γ following from eq. (39) is the lower bound

$$\begin{aligned} \cot \gamma &< \frac{-I - L\,|C|}{F} \\ &= \frac{-\sqrt{1 - C^2 - S^2}\sin 2\tilde{\beta} - S \cos 2\tilde{\beta} - L\,|C|}{1 - \sqrt{1 - C^2 - S^2}\cos 2\tilde{\beta} + S \sin 2\tilde{\beta}}. \end{aligned} \tag{42}$$

This bound depends on the measured values of C, S, $\sin 2\tilde{\beta}$ and, besides, since $\cos 2\tilde{\beta}$ is positive in the SM, it depends on the sign of $\sqrt{1 - C^2 - S^2}$.

4 Application to the Belle results

The BS bound applies to the situation where S has been measured while C remains unknown but, in reality, both the BABAR and Belle Collaborations are able to measure S and C simultaneously and with comparable accuracy. Early results made public by BABAR [6] are

$$\begin{aligned} S &\in [-0.54,\ 0.58]\,, \\ C &\in [-0.72,\ 0.12] \end{aligned} \tag{43}$$

at 90% Confidence Level (C.L.). In this section I shall rather use the latest results by the Belle Collaboration [4]. Belle measures S and C to be both negative and not satisfying the constraint $S^2 + C^2 \leq 1$; enforcing the latest constraint, the Belle Collaboration has presented the allowed regions for C and S displayed in fig. 1. The point $C = 0$, $S = -\sin 2\beta$ is disallowed at 99.9157 C.L., and therefore setting a BS lower bound on γ is possible. Assuming the SM, the lower bound on γ that I shall consider is given by the inequality (42), where $\sqrt{1 - C^2 - S^2}$ may be either positive or negative—we must use, for each pair of values for S and C, the sign of $\sqrt{1 - C^2 - S^2}$ yielding the less stringent bound. I shall assume fixed values for $\sin 2\tilde{\beta} = 0.736$ and $\cos 2\tilde{\beta} = +\sqrt{1 - 0.736^2}$. For L I shall take the four values $L = 0$—the case relevant for the BS bound, where $\mathrm{Re}\, z > 0$, but no lower bound on $|\cot \arg z|$, is assumed—and $L = \cot 0.9$, $\cot 0.65$, and $\cot 0.4$, corresponding to the 3σ, 2σ, and 1σ bounds, respectively, following from eq. (9).

I performed scans of the allowed regions in the (C, S) plane advocated by the Belle Collaboration. For each value of the pair (C, S), and for each value of L, I computed the corresponding lower bound on γ. The results are the following. If one takes the 68.3% C.L. domain of Belle, then $\gamma > 21.8°$ if $L = 0$, $\gamma > 42.3°$ if $L = \cot 0.9$, $\gamma > 58.3°$ if $L = \cot 0.65$, and $\gamma > 93.6°$ if $L = \cot 0.4$. When one uses the the region allowed by Belle at 95.45% C.L., one obtains $\gamma > 12.3°$ if $L = 0$, $\gamma > 24.1°$ if $L = \cot 0.9$, $\gamma > 33.9°$ if $L = \cot 0.65$, and $\gamma > 53.7°$ if $L = \cot 0.4$. Considering at last the 99.73% C.L. limits of Belle, one gets $\gamma > 3.6°$ if $L = 0$, $\gamma > 6.6°$ if $L = \cot 0.9$, $\gamma > 8.9°$ if $L = \cot 0.65$, and $\gamma > 12.5°$ if $L = \cot 0.4$; these very loose bounds reflect the proximity to this region of the point $C = 0$, $S = -\sin 2\beta$, for which no lower bound on γ is possible any more.

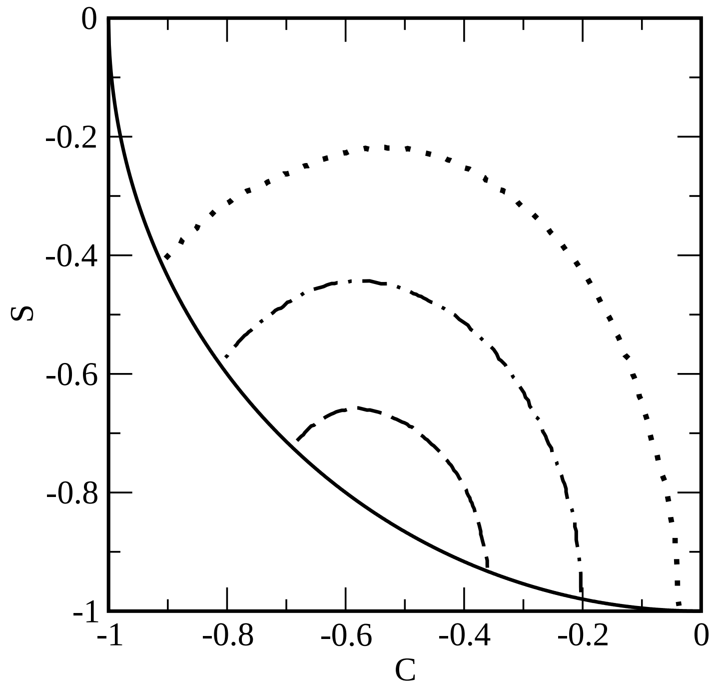

Figure 1: The latest results of the Belle Collaboration for S and C. The full line bounds the circle defined by the condition $C^2+S^2 \leq 1$. Within that circle, the dashed line bounds the region allowed by Belle at 68.3% C.L., the dot-dashed line bounds the region allowed at 95.45% C.L., and the dotted line bounds the region allowed at 99.73% C.L.

It is evident from the results above that assuming $|\cot \arg z| > L$, with a non-zero L, may greatly improve the lower bound on γ that one obtains from the BS condition $\mathrm{Re}\, z > 0$ alone.

5 Conclusions

I have shown in this Letter that the Buchalla–Safir lower bound on γ is a purely algebraic consequence of the assumption $\mathrm{Re}\, z > 0$; the latter assumption follows from a computation of z within the Standard Model but, after that computation, the derivation of the BS bound itself requires no physics. I have emphasized that a better lower bound on γ may be obtained if one considers that, besides S, also C is known. I have improved the BS bound by assuming, above and beyond $\mathrm{Re}\, z > 0$, a lower bound on $|\cot \arg z|$. I have emphasized the fact that the presence, within the experimentally allowed region, of the point $(S, C) = (-\sin 2\beta, 0)$, prevents one from putting a lower bound on γ. I have applied the derived bounds to the (S, C) domains allowed by the most recent results made public by the Belle Collaboration.

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References

[1] For an introduction to CP violation in the B system see G. C. Branco, L. Lavoura, and J. P. Silva, *CP violation* (Oxford, England: Oxford University Press, 1999).

[2] T. E. Browder, "Results on the CKM angle ϕ_1 (β)," hep-ex/0312024, to appear in the Proceedings of the 2003 Lepton–Photon Conference.

[3] G. Buchalla and A. S. Safir, "Model-independent bound on the unitarity triangle from CP violation in $B \to \pi^+\pi^-$ and $B \to \psi K_S$," hep-ph/0310218.

[4] K. Abe *et al.* (The Belle Collaboration), "Observation of large CP violation and evidence for direct CP violation in $B^0 \to \pi^+\pi^-$ decays," hep-ex/0401029.

[5] F. J. Botella and J. P. Silva, "Bounds on γ from CP-violation measurements in $B \to \pi^+\pi^-$ and $B \to \psi K_S$," hep-ph/0312337.

[6] B. Aubert *et al.* (BABAR Collaboration), "Measurements of branching fractions and CP-violating asymmetries in $B^0 \to \pi^+\pi^-$, $K^+\pi^-$, K^+K^- decays," *Physical Review Letters* **89**, 281802 (2002).